THE MASLOV CLASS OF THE LAGRANGE SURFACES
AND GROMOV'S PSEUDO-HOLOMORPHIC CURVES

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ABSTRACT. For an immersed Lagrange submanifold $W \subset T^*X$, one can define a nonnegative integer topologic invariant $m(W)$ such that the image of $H_1(W; \mathbf{Z})$ under the Maslov class is equal to $m(W) \cdot \mathbf{Z}$. In this paper, the value of $m(W)$ is calculated for the case of a two-dimensional oriented manifold X with the universal cover homeomorphic to $\mathbf{R}^2$ and an embedded Lagrange torus W. It is proved that if $X = \mathbf{T}^2$ and W is homologic to the zero section, then $m(W) = 0$. In all the other cases $m(W) = 2$. The last result is true also for a wide class of oriented properly embedded Lagrange surfaces in $T^*\mathbf{R}^2$. The proof is based on the Gromov's theory of pseudo-holomorphic curves. Some applications to the hamiltonian mechanics are mentioned.

1. INTRODUCTION

Many actual problems of the global symplectic topology lead to studying different invariants of a Lagrange submanifold in an ambient symplectic manifold. Following are two such problems:

(a) [Ar] What invariants can distinguish different components of the set of the Lagrange embeddings?
(b) [Au] What components of the set of the Lagrange immersions contain a Lagrange embedding?

Let $V = T^*X^n$ be a cotangent bundle with the natural symplectic structure. The simplest topological invariant of an immersed Lagrange submanifold $W \subset V$ is a nonnegative integer $m(W)$ which is the generator of the subgroup $\mu(H_1(W; \mathbf{Z})) \subset \mathbf{Z}$, where $\mu \in H^1(W; \mathbf{Z})$ is the Maslov class. In the present paper we consider the case of an oriented nonelliptic surface X (i.e., the universal cover of X is homeomorphic to $\mathbf{R}^2$) and calculate $m(W)$ for any embedded Lagrange torus $W \subset T^*X$. We prove that if $X = \mathbf{T}^2$ and W is homologic to the zero section then $m(W) = 0$. In all the other cases $m(W) = 2$. Moreover we generalise this result on the wide class of properly embedded (in general, noncompact) Lagrange surfaces in $T^*\mathbf{R}^2 = \mathbf{C}^2$ (see Theorems 2.1, 2.4 below).

Note that, in the case $V = \mathbf{C}^2$, the Maslov class is well defined for the wider set of the immersed totally real surfaces. It follows from our results that

if $m(W) \neq 2$ for a totally real torus $W \subset V$, then W cannot be deformed to a Lagrange embedding (cf. question (b) above). However, for every even integer $k \geq 0$, there exists an immersed Lagrange torus W_k' and an embedded totally real torus W_k'' with $m(W_k') = m(W_k'') = k$ (see [P1] for the construction of W_k''). Note that the invariants associated with the Maslov class cannot distinguish different components of the set of the Lagrange embeddings (cf. the problem (a) above). Maybe it is a hint that there is only one component.

It was M. Audin who first raised a question about the Maslov class nontriviality [Au]. According to her hypothesis, $m(W) = 2$ for any embedded Lagrange torus in $\mathbf{C}^n$. Some steps forward were made in [BP, P2, Bi] for strongly optical Lagrange tori which usually appear in mechanics and optics (see Corollary 2.5 and the discussion below).

When the proof of the main results of the present paper was finished I received the preprint by C. Viterbo [Vi]. He proves the following: (a) $m(W) \in [2; n+1]$ for an embedded Lagrange torus $W \subset \mathbf{C}^n$; (b) $m(W) = 2$ for an embedding $W \hookrightarrow \mathbf{C}^n$ if there exists a Riemannian metric of the negative sectional curvature on W. Viterbo's proof is based on the detailed studying of periodic solutions of the special Hamiltonian system associated with the Lagrange torus. Our proof is quite different and is based on the Gromov's theory of pseudo-holomorphic curves [Gr].

The present paper is a continuation of [P3]. I am deeply grateful to V. I. Arnold, S. L. Tabachnikov and O. Ya. Viro for useful discussions. I would like to thank A. B. Givental for many helpful suggestions and explaining to me the crucial inequality in the proof of Corollary 3.2.

2. Main results

We assume that the complex space $\mathbf{C}^2$ is endowed with the standard symplectic form ω, complex structure J_0 and Euclidean metric d. Denote by $l(\cdot, \cdot)$ the coefficient of enlacement.

Theorem 2.1. Let $W \subset \mathbf{C}^2$ be a properly embedded orientable Lagrange submanifold with the following properties:

(a) The injectivity radii of the induced Riemannian metric d_W on W and of the normal exponential map of W are bounded away from zero; (b) there exist positive numbers c and x such that, for every two points w_1, w_2 in W, $d_W(w_1, w_2) < cd(w_1, w_2)$ holds when $d(w_1, w_2) < x$; (c) there exists a nonzero linear holomorphic function $h: \mathbf{C}^2 \to \mathbf{C}$ such that $h(W)$ is a bounded subset in $\mathbf{C}$.

Then, for every $\delta > 0$, there exists a cycle $\gamma \in H_1(W; \mathbf{Z})$ such that:

(1) $\mu(\gamma) = 2$ where μ is the Maslov class (i.e. $m(W) = 2$);
(2) the symplectic area $\omega(\gamma)$ of γ is positive;
(3) $l(\gamma, L) \geq 0$ for every properly embedded holomorphic surface $L \subset \mathbf{C}^2$ with $d(L, W) > \delta$.

Let us give some examples of the Lagrange surfaces in $\mathbf{C}^2$ with the properties (a)–(c).

Example 2.2. An embedded Lagrange torus.

Example 2.3. Let $\mathbf{C}^2 = T^*\mathbf{R}^2 \to T^*(\mathbf{R} \times S^1)$ be a universal covering. Let $\widetilde{W} \subset T^*(\mathbf{R} \times S^1)$ be an embedded Lagrange torus such that the embedding induces an epimorphism of the fundamental groups. Then its lift $W \subset \mathbf{C}^2$ is a noncompact surface which obviously satisfies (a)–(c).

Theorem 2.4. *Let X be an oriented nonelliptic surface (i.e., its universal cover is homeomorphic to $\mathbf{R}^2$). Let $W \subset T^*X$ be an embedded Lagrange torus.*

(a) *If $X = \mathbf{T}^2$ and W is homologic to the zero section, then the Maslov class μ vanishes: $m(W) = 0$.*

(b) *In all the other cases (i.e., either $X \neq \mathbf{T}^2$ or $X = \mathbf{T}^2$ and W is not homologic to the zero section) there exists a cycle $\gamma \in H_1(W; \mathbf{Z})$ with $\mu(\gamma) = 2$ (i.e., $m(W) = 2$).*

Mention some applications of these results to mechanics. Let X be as in Theorem 2.4 and $H(p, q)$ be a smooth hamiltonian function on T^*X which is strictly convex on each fiber $\{q = \text{const}\}$ and assume its minimum value at zero. Assume that $\max H(0, q) < \infty$. Let $W \subset \{H = h\}$ be an embedded Lagrange torus (this means that W is invariant under the hamiltonian flow), where $h > \max H(0, q)$. Let's assume that W does not contain closed trajectories of the flow. Recall from [BP] that the set of the critical points of the natural projection $\pi \colon W \to \mathbf{T}^2$ is a closed submanifold consisting of smooth incontractible homotopic curves which are transverse to the hamiltonian field. Let us call them critical circles. Denote by $k(W)$ a number of the critical circles.

Corollary 2.5. (a) *(A generalisation of the second Birkhoff theorem.) If $X = \mathbf{T}^2$ and W is homologic to the zero section, then $k(W) = 0$, i.e., W is a section of the cotangent bundle.*

(b) *If either $X \neq \mathbf{T}^2$ or $X = \mathbf{T}^2$ and W is not homologic to the zero section, then $k(W) = 2$.*

Corollary 2.5(a) was first proved in [BP]. Corollary 2.5(b) had been first proved in [Bi] under some topological restrictions on the embedding $W \hookrightarrow \{H = h\}$ and, afterwards, M. Bialy proved the general case (private communication). However, in all cases it was assumed that the hamiltonian function H is symmetric: $H(p, q) = H(-p, q)$. Now we are free from this restriction.

Proof of Corollary 2.5. Let $\gamma \in H_1(W; \mathbf{Z})$ be the class of the critical circle cooriented by the hamiltonian field. Then the Maslov class of W is dual to the cycle $k(W) \cdot \gamma$ [BP]. Thus each corollary follows from Theorem 2.4. □

We will complete this section by deducing Theorem 2.4 from Theorem 2.1.

Proof of Theorem 2.4. (a) Note that the natural projection $\pi\colon T^*\mathbf{T}^2 \supset W \to \mathbf{T}^2$ induces an isomorphism $\pi_\#\colon H_1(W;\mathbf{Z}) \to H_1(\mathbf{T}^2,\mathbf{Z})$. Suppose that the Maslov class $\mu \in H^1(W;\mathbf{Z})$ is not zero. Then there exists a basis α, β in $H_1(W;\mathbf{Z})$ such that $\mu(\alpha) \geq 2$, $\mu(\beta) \geq 2$. We may assume that $\pi_\#(\alpha) = [S^1 \times \{\cdot\}]$, $\pi_\#(\beta) = [\{\cdot\} \times S^1]$, where $\mathbf{T}^2 = S^1 \times S^1$. Let $(p_1, q_1; p_2, q_2)$ be the canonic coordinates in $T^*\mathbf{T}^2 = T^*S^1 \times T^*S^1$. Assume that the torus W lies in $M = \{|p_1| < c\} \times \{|p_2| < c\}$ for some $c > 0$. Each annulus $\{|p_i| < c\}$, $i = 1, 2$, is symplectomorphic to an annulus $K_i = \{c_i < |z_i| < c_2\} \subset \mathbf{C}^1(z_i)$ for some $c_2 > c_1 > 0$. Let W_1 be the image of W under the map $f\colon W \hookrightarrow M \to K_1 \times K_2 \to \mathbf{C}^1(z_1) \times \mathbf{C}^1(z_2) = \mathbf{C}^2(z_1, z_2)$. Let $\mu_1 \in H^1(W_1;\mathbf{Z})$ be the Maslov class of the embedded Lagrange torus W_1 in $\mathbf{C}^2$. Denote $\alpha_1 = f_\#(\alpha)$, $\beta_1 = f_\#(\beta)$. It is easy to see that $\mu_1(\alpha_1) - \mu(\alpha) = \mu_1(\beta_1) - \mu(\beta) = 2$. Then $\mu_1(\alpha_1) \geq 4$, $\mu_1(\beta_1) \geq 4$. Denote $L_j = \{z_j = 0\}$, $j = 1, 2$. It is easy to check that $l(\alpha_1, L_1) = l(\beta_1, L_2) = 1$, $l(\alpha_1, L_2) = l(\beta_1, L_1) = 0$ (where l is the coefficient of enlacement). Now Theorem 2.1 gives us the cycle $\gamma = m\alpha_1 + n\beta_1$ with $\mu_1(\gamma) = 2$, $l(\gamma, L_j) \geq 0$ for $j = 1, 2$. This means that $m \geq 0$, $n \geq 0$ and $m\mu_1(\alpha_1) + n\mu_1(\beta_1) = 2$, where $\mu_1(\alpha_1) \geq 4$, $\mu_1(\beta_1) \geq 4$. This is obviously impossible. The contradiction shows that $\mu \equiv 0$.

(b) Note that it is sufficient to consider the case of a closed surface X. Let $\pi\colon W \to X$ be the natural projection. Then the induced homomorphism $\varphi\colon \pi_1(W) \to \pi_1(X)$ has a nontrivial kernel. Indeed, it is obvious when X has genus greater than 1. In the case $X = \mathbf{T}^2$ it follows from [Ar, §9].

Let $\mathscr{U} \approx \mathbf{R}^2$ be the universal cover of X. If $\varphi(\pi_1(W)) = 0$, then consider the lift $\widetilde{W} \subset T^*\mathscr{U}$. Obviously $\widetilde{W} \approx \mathbf{T}^2$, and the theorem follows from Theorem 2.1 and Example 2.2.

Now suppose that $\varphi(\pi_1(W)) \neq 0$. Then $\varphi(\pi_1(W))$ is a cyclic subgroup of $\pi_1(X)$, say B. Let B' be a cyclic subgroup of the automorphisms of $\mathscr{U}$ which corresponds to B, $p\colon \mathscr{U} \to \mathscr{U}/B'$—the natural projection. It is easy to see that $\mathscr{U}/B'$ is diffeomorphic to $\mathbf{R} \times S^1$. Let $\hat{p}\colon T^*\mathscr{U} \to T^*(\mathscr{U}/B')$ be the fiber-preserved cover associated with p. Choose the lift $\widetilde{W} \subset T^*\mathscr{U}$ which corresponds to B'. Then $\hat{p}(\widetilde{W}) \subset T^*(\mathbf{R} \times S^1)$ is an embedded Lagrange torus and the theorem follows from Theorem 2.1 and Example 2.3. □

3. Proof of Theorem 2.1

Denote $V = \mathbf{C}^2$.

After a slight perturbation of W we may assume that there exists a neighborhood $\mathscr{U} \supset W$ and a J_0-antiholomorphic involution $\tau\colon \mathscr{U} \to \mathscr{U}$ such that $\operatorname{Fix}\tau = W$ and $d(v, W) < \delta/2$ for every $v \in \mathscr{U}$. Let $\mathscr{J}$ be the space of C^∞-almost complex structures J on V such that:

(1) $J(v) = J_0(v)$ when $d(v, W) > \delta$;
(2) τ is J-antiholomorphic;

(3) ω tames J, i.e., $\omega(\xi, J\xi) \geq \mathrm{const}\, |\xi|^2 \quad \forall \xi \in TV$;

(4) ω calibrates J, i.e., $\omega(\xi, \eta) = \omega(J\xi, J\eta) \quad \forall \xi, \eta \in TV$;

(5) $\|J\|_\varepsilon < \infty$ where $\|\ \|_\varepsilon$ is the Floer norm [F1,§5].

For every $\alpha \in H_2(V, W; \mathbf{Z}) \backslash \{0\}$, define the space $\mathscr{F}_\alpha$ of $C^{r+\nu}$-Hölder maps $f: (D^2, \partial D^2) \to (V, W)$ with $r \geq 2$, $\nu \in (0, 1)$, $[f] = \alpha$. Moreover, we demand that, for every $f \in \mathscr{F}_\alpha$, there exists a $z \in \partial D^2$ such that $f^{-1}(f(z)) = z$, $Df(z) \neq 0$ (cf. [McD, §4]). Set $M_\alpha = \{(f, J) \in \mathscr{F}_\alpha \times \mathscr{J} | \bar{\partial}_J f = 0\}$, $\bar{\partial}_J f = DF + J \circ Df \circ i$.

Proposition 3.1. *M_α is a Banach submanifold in $\mathscr{F}_\alpha \times \mathscr{J}$. The natural projection $P: M_\alpha \to \mathscr{J}$ is a Fredholm map with the index $\mathrm{Index}\, P = \mu(\hat{\alpha}) + 2$, where $\hat{\alpha}$ is the image of α in $H_1(W; \mathbf{Z})$.*

The proof is similar to [F1, §5; McD, §4]. See the index calculation in [Gr, 2.1.D].

Set $M_\alpha(J) = P^{-1}(J) \cap M_\alpha$.

Corollary 3.2. *Let $J \in \mathscr{J}$ be a regular value of P. If $M_\alpha(J) \neq \varnothing$ then $\mu(\hat{\alpha}) \geq 2$.*

Proof. The group $\mathscr{D}$ of the conformal automorphisms of D^2 acts freely on $M_\alpha(J)$. Then $\dim M_\alpha(J) = \mu(\hat{\alpha}) + 2 \geq \dim \mathscr{D} = 3$ and $\mu(\hat{\alpha}) \geq 1$. Note that, for orientable Lagrange manifolds, $\mu(H_1(W; \mathbf{Z})) \subset 2\mathbf{Z}$. Then $\mu(\hat{\alpha}) \geq 2$. $\square$

Let $\mathscr{G}_J(J \in \mathscr{J})$ be the space of C^∞-sections of the bundle $\overline{\mathrm{End}}_{(i, J)}(TC, TV) \to \mathbf{C} \times V$ of antiholomorphic endomorphisms. We demand that every $g \in \mathscr{G}_J$ is bounded in the Floer norm $\|\ \|_\varepsilon$ [F1, §5].

Proposition 3.3. *There exists a neighborhood $\mathscr{J}_0 \subset \mathscr{J}$ of J_0 with the following property: for every $J \in \mathscr{J}_0$, there is an open set $\mathscr{G}'_J \subset \mathscr{G}_J$ such that the equation $\bar{\partial}_J f = g$ has no solutions on the set $\{f \in C^{r+\nu}((D^2, \partial D^2), (V, W)), [f] = 0\}$, for every $g \in \mathscr{G}'_J$.*

The proof is based on condition (c) of the theorem and is similar to [Gr, 2.3.B1] (see Appendix for the details).

Fix a regular value $J \in \mathscr{J}_0$ of the map P. Set $V_1 = \mathbf{C} \times V$, $W_1 = \partial D^2 \times W$. For every $g \in \mathscr{G}_J$, define an almost complex structure J_g on V_1: $J_g(\xi \oplus \eta) = i\xi \oplus (g\xi + J\eta)$, where $\xi \oplus \eta \in TC \oplus TV$.

Define the space Φ_β of the $C^{r+\nu}$-maps $\varphi: (D^2, \partial D^2) \to (V_1, W_1)$, $\varphi(z) = (z, \hat{\varphi}(z))$, where $[\hat{\varphi}] = \beta \in H_2(V, W; \mathbf{Z})$. Let $N_\beta = \{(\varphi, g) \in \Phi_\beta \times \mathscr{G}_J | \bar{\partial}_{J_g} \varphi = 0\}$. Note that $\bar{\partial}_{J_g} \varphi = 0 \Leftrightarrow \bar{\partial}_J \hat{\varphi} = g$ [Gr, 1.4.C]. The following fact is similar to Proposition 3.1.

Proposition 3.4. *N_β is a Banach submanifold in $\Phi_\beta \times \mathscr{G}_J$. The natural projection $P_1: N_\beta \to \mathscr{G}_J$ is a Fredholm map with index $\mathrm{Index}\, P_1 = \mu(\hat{\beta}) + 2$, where $\hat{\beta}$ is the image of β in $H_1(W; \mathbf{Z})$.*

Let g_0, g_1 be regular values of P_1. Let $\gamma: [0, 1] \to \mathscr{G}_J$ be a path such that $\gamma(0) = g_0$, $\gamma(1) = g_1$ and $P_1 \pitchfork \gamma$ (for all β).

Set $Y_\beta = P_1^{-1}(\gamma) \cap N_\beta$.

Corollary 3.5. *If $Y_\beta \neq \varnothing$ then $\mu(\hat{\beta}) \geq -3$.*

Proof. $\dim Y_\beta = \operatorname{Index} P_1 + 1 = \mu(\hat{\beta}) + 3 \geq 0$. $\square$

Assume now that $g_1 \in \mathscr{G}_J'$ and $\|g_0\|_\varepsilon$ is sufficiently small.

Proposition 3.6. *There exist the classes $\beta, \alpha_1, \ldots, \alpha_k \in H_2(V, W; \mathbf{Z})$ such that $\alpha_i \neq 0$, $M_{\alpha_i}(J) \neq \varnothing$ $(i = 1, \ldots, k)$, $Y_\beta \neq \varnothing$ and $\beta + \sum_1^k \alpha_i = 0$.*

Proof of Theorem 2.1. Corollaries 3.2 and 3.5 give us that $\mu(\hat{\alpha}_i) \geq 2$, $\mu(\hat{\beta}) \geq -3$, $\mu(\hat{\beta}) + \sum_1^k \mu(\hat{\alpha}_i) = 0$. Then the only possibility is that $k = 1$, $\mu(\hat{\beta}) = -2$, $\mu(\hat{\alpha}_1) = 2$. Moreover, there exists a J-holomorphic curve $f \in M_{\alpha_1}(J)$. Then $\omega(\hat{\alpha}_1) > 0$. The structure J coincides with J_0 on the set $\{v \in V | d(v, W) > \delta\}$. Then the intersection index of f with any holomorphic surface L, such that $d(L, W) > \delta$, is nonnegative. This gives us the last assertion of the theorem. $\square$

Proof of Proposition 3.6. In fact this is proved in [Gr, 2.3.B]. Nevertheless, we will outline the proof in order to explain the conditions (a), (b) of Theorem 2.1. Define a map $A: Y_0 \to W$, $(\varphi, g) \mapsto \hat{\varphi}(1)$. Fix a regular value, say w of A. Then $X = A^{-1}(w)$ is a nonempty manifold with $\dim X = 1$. It is easy to see that if X is compact then $\#\{P_1^{-1}(g_1) \cap X\} = 1 \pmod 2$, i.e., there exists a map $\hat{\varphi}: (D^2, \partial D^2) \to (V, W)$ such that $[\hat{\varphi}] = 0$ and $\overline{\partial}_J \hat{\varphi} = g_1$. But this contradicts Proposition 3.3.

Then X is noncompact. Note that it follows from the conditions (a), (b) that all the maps $\hat{\varphi}: (D^2, \partial D^2) \to (V, W)$ with $[\hat{\varphi}] = 0$, $\hat{\varphi}(1) = w$, $\overline{\partial}_J \hat{\varphi} \in \gamma$ are bounded. Now the Gromov compactness theorem [Gr, 1.5.B; Pa] gives us a J_g-cusp-curve with the boundary on W_1 which is the union of a curve $\varphi \in Y_\beta$ and some nonconstant curves $f_1, \ldots, f_k$. The last lie in the fibers $\{z_j\} \times V$. Note that the restriction of J_g on any such fiber coincides with J. Then $f_j(z) = (z_j, \hat{f}_j(z))$, where $\hat{f}_j \in M_{\alpha_j}(J)$ for some $\alpha_j \neq 0$. Moreover, this cuspcurve is a limit for some sequence from Y_0. Then $\beta + \sum \alpha_j = 0$. $\square$

Note 3.7. In [Gr, 2.3.B] another condition was used instead of our (a) in order to estimate the diameter of any J_g-curve with the boundary on W_1. However, it is easy to check that the conditions (a), (b) are sufficient for our purposes (the proof is similar to [Si, III.4]).

Note 3.8. It does not follow immediately from the compactness theorem that the pseudo-holomorphic curves $\hat{f}_j$ above are injective at some point of the boundary (i.e., $\hat{f}_j \in \mathscr{F}_{\alpha_j}$). However, if some $\hat{f}_j$ does not lie in $\mathscr{F}_{\alpha_j}$, then it

is multiply covered. This means that $\hat{f}_j(z) = f_j'(a_j(z))$, where $f_j' \in M_{\alpha_j'}(J)$ and $a_j : D^2 \to D^2$ is a proper holomorphic map with degree $p > 1$. Obviously $\alpha_j = p\alpha_j'$. The proof of this assertion is similar to [McD, Lemma 4.4(i)].

APPENDIX. PROOF OF PROPOSITION 3.3

Let (v_1, v_2) and $z = x + iy$ be the complex coordinates on $V = \mathbf{C}^2$ and $\mathbf{C}^1$ respectively. Indentify every tangent space $T_{(\cdot)}V$ with $\mathbf{C}^2(v_1, v_2)$. For any $g \in \mathscr{G}_J$, set $\tilde{g}(z, v) = g(z, v)\frac{\partial}{\partial x} = (g_1(z, v), g_2(z, v))$. Without loss of generality, we may assume that the map $h : V \to \mathbf{C}$ (see the condition (c) of Theorem 2.1) is equal to $(v_1, v_2) \mapsto v_1$. Due to our assumption, $|h(w)| < c_h$ $\forall w \in W$, for some $c_h > 0$. Recall that the symplectic form ω tames and calibrates every almost complex structure $J \in \mathscr{J}$. Then one can define a Riemannian norm $|\xi|_J^2 = \omega(\xi, J\xi)$, where $\xi \in T_{(\cdot)}V$, which coincides with the Euclidian one for $J = J_0$. We will use the following notations for L^2-norms:

$$\|\psi\|_2 = \left(\int_{D^2} |\psi|^2\, dx\, dy\right)^{1/2},$$

where $\psi : D^2 \to \mathbf{C}$;

$$\|\psi\|_2 = \left(\int_{D^2} \sum_1^p |\psi_j|^2\, dx\, dy\right)^{1/2}, \qquad \|\psi\|_{2,J} = \left(\int_{D^2} \sum_1^p |\psi_j|_J^2\, dx\, dy\right)^{1/2},$$

where $\psi : D^2 \to (\mathbf{C}^2)^p$, $p = 1, 2$;

$$\|g\|_2 = \left(\int_{D^2} \sup_v |g_1(z, v)|^2 + \sup_v |g_2(z, v)|^2\, dx\, dy\right)^{1/2},$$

where $g \in \mathscr{G}_J$.

Denote by $c_1, c_2, \ldots, \varepsilon, A$, some positive constants which will be chosen below. Define the sets

$$\mathscr{J}_0 = \{J \in \mathscr{J} \mid |J\xi - J_0\xi| < \varepsilon|\xi|\ \forall \xi \in T_{(\cdot)}V,\ \xi \neq 0\},$$

$$\mathscr{G}_J' = \{g \in \mathscr{G}_J \mid \mathrm{Reg}_1(z, v) > A,\ \|g\|_2 < c_1 A\}.$$

These sets are nonempty for any $\varepsilon > 0$ and c_1 sufficiently large. We claim that if ε is sufficiently small and A is sufficiently large, then the equation $\overline{\partial}_J f = g$ has no solutions on the set $\{f \in C^{r+\nu}((D^2, \partial D^2), (V, W)),\ [f] = 0\}$ for any $J \in \mathscr{J}_0$, $g \in \mathscr{G}_J'$.

Suppose that $\overline{\partial}_J f = g$. Then

(1) $g_1(z, f(z)) = \partial f_1/\partial x + i\partial f_1/\partial y + R(z)$,   where $|R| < \varepsilon|\nabla f|$.

Notice that $\|\tilde{g}(z, f(z))\|_{2,J} = \|\nabla f\|_{2,J}$, since $\int_{D^2} f^*\omega = 0$ (where we use that W is Lagrange and $[f] = 0$). Thus

(2) $\|\nabla f\|_2 \leq c_2\|g\|_2 < c_2 c_1 A$.

From (1) we obtain that

$$(3) \quad \left| \operatorname{Re} \int_{D^2} g_1(z, f(z))\, dx\, dy \right| \leq \left| \operatorname{Re} \int_{D^2} (\partial f_1/\partial x + i \partial f_1/\partial y)\, dx\, dy \right|$$
$$+ \left| \operatorname{Re} \int_{D^2} R\, dx\, dy \right|.$$

Now use that $h(W)$ is bounded:

$$(4) \quad \left| \operatorname{Re} \int_{D^2} (\partial f_1/\partial x + i \partial f_1/\partial y)\, dx\, dy \right| \leq c_3 \max_{\partial D^2} |f_1|$$
$$\leq c_3 \max_W |h| \leq c_3 c_h = c_4.$$

From (2) we conclude that

$$(5) \quad \left| \operatorname{Re} \int_{D^2} R\, dx\, dy \right| \leq c_5 \|R\|_2 \leq c_5 \varepsilon \|\nabla f\|_2 \leq \varepsilon c_6 A.$$

Now, (5), (4) and (3) give that $2\pi A \leq c_4 + \varepsilon c_6 A$. It is clear now that if $\varepsilon < c_6^{-1}$, $A > c_4/(2\pi - 1)$, then the equation $\overline{\partial}_J f = g$ has no solutions. $\quad \square$

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